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Product of Locally Nilpotent Groups

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ABSTRACT: A group is said to be locally nilpotent if every finitely generated subgroup of the group is nilpotent. In this paper we show that if the group G=AB=AK=BK be the product of three locally nilpotent subgroups A,B, and K, where K is normal in G. And G has finite abelian section rank, then G is locally nilpotent and hence hypercentral.

***Keywords*:** min-by-max, minimal condition, maximal condition, finite residual group .

INTRODUCTION

In 1968 N.F. Sesekin (see [19]) proved that a product of two abelian subgroups with minimal condition satisfies also the minimal condition . He and Amberg independently obtained a similar result for the maximal condition around 1972. Moreover, a little later the proved that a soluble product of two nilpotent subgroups with maximal condition likewise satisfies the maximal condition, and its Fitting subgroups inherits the factorization. Subsequently in his Habilitationsschrift (1973) he started a more systematic investigation of the following general question. Given a (soluble) product G of two subgroups A and B satisfying a certain finiteness condition $\mathfrak{X}$, when does G have the same finiteness condition $\mathfrak{X}$?(see [20])

For almost all finiteness conditions this question has meanwhile been solved. Roughly speaking, the answer is 'yes' for soluble (and even for soluble-by-finite) groups. This combines theorems of B. Amberg (see [1], [2],[3],[4] and [6]) , N.S. Chernikov (see [5]), S. Franciosi, F. de Giovanni (see [3],[6]), O.H.Kegel (see [8]), J.C.Lennox (see [12]) , D.J.S. Robinson(see [9] and [12]), J.E. Roseblade(see [13]), Y.P.Sysak(see [19] and[20]), J.S. Wilson(see [23]), and D.I.Zaitsev(see [11] and [18]).

Now, in this paper, we study the locally nilpotent subgroups G and its relations, and the end we prove that if the group G=AB=AK=BK be the product of three locally nilpotent subgroups A,B, and K, where K is normal in G. And G has finite abelian section rank, then G is locally nilpotent and hence hypercentral.

2. Priliminaries: (elementary properties and theorems.)
In this chapter we introduce the elementary properties, Lemma and theorems.

2.1. Lemma: Let the finite group G=AB be the product of two subgroups A and B. If A,B, and G are D_π *- group,* for a set π of primes, then there exist Hall π-subgroups A_0 of A and B_0 of B such that A_0B_0 is a Hall π-subgroups of G.

Proof: Let A_1, B_1, and G_1 be Hall π-subgroups of A, B, and G, respectively. Since G is a D_π *- group,* there exist elements x and y such that A_1^x *and* B_1^y are both contained in G_1. It follows from Lemma 2.4 that $A^x = A^z$ and $B^y = B^z$ for some z in G. Thus $A_0 = A_1^{xz^{-1}}$ and $B_0 = B_1^{yz^{-1}}$ are Hall π-subgroups of A and B, respectively, which are both contained in $G_0 = G_1^{yz^{-1}}$. Clearly the order of $A_0 \cap B_0$ is bounded by the maximum π-divisor n of the order of $A \cap B$ since $|G| = \frac{|A|.|B|}{|A \cap B|}$, It follows that $|G_0| = \frac{|A_0|.|B_0|}{n} \le \frac{|A_0|.|B_0|}{|A_0 \cap B_0|} = |A_0B_0|$. Therefore $A_0B_0=G_0$ is a Hall π-subgroup of G.

2.2.Corollary: Let the finite group G=AB be the product of two subgroups A and B .Then for each prime p there exist Sylow p-subgroups A_0 of A and B_0 of B such that A_0B_0 is a Sylow p-subgroup of G.

Proof: See [5]

2. 8. Corollary : Let the finite group G=AB=AK=BK be the product of three nilpotent subgroups, A,B, and K, where K is normal in G. Then G is nilpotent .

Proof: See([4], corollary 1.3.5)

2. 4.Lemma: Let D be a tensorial class of G-modules. Then the class $P'D$ of G-modules having an ascending series of submodules whose factors belong to D is also tensorial.

Proof: Let A and B two G-models in the class $P'D$, and consider ascending series of submodules $0=A_0 \leq A_1 \leq \ldots \leq A_\sigma = A$ and $0=B_0 \leq B_1 \leq \ldots \leq B_r = B$ with factors in D. Let T denote the tensor product $A \otimes_t B$, and for each ordinal $\mu \leq \sigma + \tau$, let T_μ be the subgroup of T generated by all $a \otimes b$, where a is in A_α, b is in B_β and $\alpha + \beta \leq \mu$. Clearly T_μ is a G-submodule of T, and we have the following ascending series of submodules $0=T_0=T_1 \leq T_2 \leq \ldots \leq T_{b+r}=T$.(6.3) Let μ be an ordinal such that $T_\mu < T_{\mu+1}$. Then there exist ordinals $\alpha < b$ and $\beta < \tau$ such that $(\alpha+1)+(\beta+1)=\mu+1$. The map $(a+A_\alpha, b+B_\alpha) \mapsto a \otimes b + T_\mu$, where a is in $A_{\alpha+1}$ and b is in $B_{\beta+1}$, is well defined and bilinear, and hence induces a G-homomophism from $(A_{\alpha+1}/A_\alpha) \otimes_Z (B_{\beta+1}/B_\beta)$ into $T_{\mu+1}/T_\mu$. It follows that there exists a G-epimorphism

$$\bigoplus_{(\alpha+1)+(\beta+1)=\mu+1} (A_{\alpha+1}/A_\alpha) \otimes_Z (B_{\beta+1}/B_\beta) \rightarrow T_{\mu+1}/T_\mu .$$

Therefore the series (6.3) can be refined to an ascending series of submodules whose factors are G-homomorphic images of certain tensor products $(A_{\alpha+1}/A_\alpha) \otimes_Z (B_{\beta+1}/B_\beta)$. Hence each G-homomorphic image of T is in the class $P'D$. The lemma is proved.

2.5. Difinition(See [15]): Recall that the Baer radical of a group G is the subgroup generated by all its abelian subnormal subgroups. In particular the Baer radical is locally nilpotent. A group is called a Baer radical.

2.6.Lemma (See 9): Let the group G=AB=AK=BK be the product of three nilpotent subgroups A,B, and K, where K is normal in G, and assume that the Baer radical of G is nilpotent. If there exists a normal subgroup N of G such that the factorizer X(N) of N in G=AB and the factor group G/N are nilpotent, the G is nilpotent.

Proof: Since G/N is nilpotent, the factorizer X(N) is subnormal in G. Therefore X(N)lies in the Baer radical L of G. Clearly G=AK=AL, so that

$$[A \cap BN, \underbrace{G, \ldots, G}_{\leftarrow r \rightarrow}] \leq [A \cap BN, \underbrace{A, \ldots, A}_{\leftarrow r \rightarrow}]L' = L'$$

for a sufficiently large integer r. Thus $(A \cap BN)L'/L'$ is contained in some term with finite ordinal type of the upper central series of G/L'. Of course, a similar statement is true for $(B \cap AN)L'/L'$,and hence NL'/L' is contained in some term with finite ordinal type of the upper central series of G/L'. Consequwntly G/L' is nilpotent. By hypothesis L is nilpotent and so by Hall's theorem G is also nilpotent (See [15]).

2.7. Difinition: A group G is called minimax if it has a series of finite length whose factors either satisfy the minimal or the maximal condition.

2.8. Theorem: Let the group G=AB=AK=BK be the product of three nilpotent subgroups A, B, and K, where K is normal in G. If K is minimax, then G is nilpotent.

Proof: Assume that the theorem is false, and among the counter-examples with K of minimal minimax rank choose a group G for which the sum of the nilpotency classes of A and B is also minimal. By Hall's theorem we may suppose that K it abelian (See [15], Part 1, Theorem 2.27). If G is finite-by-nilpotent, then $|G:Z_n(G)|$ is finite for some non-negative integer n (See [15], Part 1,Theorem 4.25). The finite factor group $G/Z_n(G)$ is nilpotent by corollary 2.3, and so G is also nilpotent. This contradiction shows that G is not finite-by-nilpotent.

Assume first that K is periodic, so that it is a Chernikov group. Hence K contains a finite G-invariant subgroup E such that K/E is radicable. Since G/E is not nilpotent, we may suppose that K is radicable. Let H be an infinite G-invariant subgroup of K. If H is properly contained in K, the factor group G/H and the factorizer X(H) of H in G are nilpotent. By Lemma 2.6(ii) the Baer radical of G is nilpotent, so that G is nilpotent by Lemma 2.5. This contradiction shows that every proper G-invariant subgroup of K is finite. Clearly the normal subgroups $A\bigcap K$ and $B\bigcap K$ of G are properly contained in K, so that $C=(A\bigcap K)(B\bigcap K)$ is a finite normal subgroup of G. Thus G/C is not nilpotent, and so we may suppose that $A\bigcap K=B\bigcap K=1.$ By Lemma 1.1.8 of [4] the normal subgroup [K,a] of G is properly contained in K, for every a in Z(A). Since K is radicable, this implies that [K,a]=1. Therefore Z(A) is contained in Z(G), and hence G/Z(G) is nilpotent. This contradiction shows that K cannot be periodic.

Let T be the subgroup of all elements of finite order of K. It follows from the first part of the proof that the factorizer X(T) of T in G=AB is nilpotent. By Lemma 2.6(ii) the Baer radical of G is nilpotent, so that G/T is not nilpotent by Lemma 2.5. Hence we may suppose that K is torsion-free. For every prime p the factor group K/K^p is finite, so that G/K^p is nilpotent. It follows that $[K,G\overset{,\dots,}{\leftarrow r\rightarrow}G]$ is contained in K^p, where r is the Prüfer rank of K. Since K is a torsion-free abelian minimax group, by Lemma 2.32 we have $\bigcap_p K^p=1.$ Consequently $[K,G\overset{,\dots,}{\leftarrow r\rightarrow}G]=1$. Hence G is nilpotent, and this last contradiction completes the proof of the theorem.

3. Main Theorem: In this chapter we prove the main theorem.

***3.1. Main Theorem*:** Let the group G=AB=AK=BK be the product of three locally nilpotent subgroups A,B, and K, where K is normal in G. If G has finite abelian section rank, then G is locally nilpotent and hence hypercentral.

Proof: Assume that the theorem is false. Among the counterexamples with minimal torsion-free rank, consider those for which the subgroup T of all elements of finite order of K is a p-group, for some prime p. Now choose a Counterexample G such that the finite residual J of T has minimal Prüfer rank.

Suppose first that K it nilpotent. Then we may assume that K is abelian by Theorem 2.4(i). As the hypercentre factor grorup $G/\bar{Z}(G)$ is not locally nilpotent, without loss of generality Z(G)=1. The intersection $A\bigcap K$ lies in the hypercentre of A, and so is also contained in $\bar{Z}(G)$. Thus $A\bigcap K=1$ and similarly $B\bigcap K=1$.

Write $\bar{G}=G/T.$ Since $\bar{A}$ and $\bar{B}$ are hypercentral, the centralizers $C_{\bar{A}}(\bar{K})$ and $C_{\bar{B}}(\bar{K})$ are contained in the hypercentre $\bar{Z}(\bar{G})$ of $\bar{G}$. As $\bar{A}\bar{Z}(\bar{G})/\bar{Z}(\bar{G})$ and $\bar{B}\bar{Z}(\bar{G})/\bar{Z}(\bar{G})$ are homomorphic images of locally nilpotent groups of automorphisms of tha torsion-free abelian group of finite Prüfer rank $\bar{K}$, they are nilpotent (See [15], Part 2, Corollary 2 to Theorem 6.32) If $\bar{Z}(\bar{G})$ is periodic, then $\bar{K}\cap\bar{Z}(\bar{G})$=1. Thus $\bar{G}/\bar{Z}(\bar{G})$ is nilpotent by Theorem 6.3.6 of [4], so that $\bar{G}$ is hypercentral. If $\bar{Z}(\bar{G})$ is not periodic, it follow by induction that $\bar{G}/\bar{Z}(\bar{G})$ is hypercentral, so that $\bar{G}$ is hypercentral also in this case . The proof can now be completed as that of Theorem 2.8.

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